Numerical Methods

Solutions to exercises

1. Find the interpolated value at $x = 2$ using Lagrange interpolation of the following support points:

   a.) $(-4, 1)$ and $(3, 2)$
   $p_0 = y_0 \frac{x-x_1}{x_0-x_1} = 1 \cdot \frac{x-3}{-4-3} = -\frac{1}{7}x + \frac{3}{7}$
   $p_1 = y_1 \frac{x-x_0}{x_1-x_0} = 2 \cdot \frac{x+4}{3+4} = -\frac{2}{7}x + \frac{8}{7}$

   $p(x) = p_0 + p_1 = \frac{1}{7}x + \frac{11}{7}$, so $p(2) = \frac{13}{7}$.

   b.) $(-2, -2)$, $(3, -4\frac{1}{2})$, and $(1, -\frac{1}{2})$
   $p_0 = y_0 \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} = -2 \cdot \frac{x-3}{-2-3} \cdot \frac{x-1}{-2-1} = -\frac{2}{15}(x-3)(x-1)$
   $p_1 = y_1 \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} = -4\frac{1}{2} \cdot \frac{x+2}{3+2} \cdot \frac{x-1}{3-1} = -\frac{9}{20}(x+2)(x-1)$
   $p_2 = y_2 \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} = -\frac{1}{2} \cdot \frac{x+2}{1+2} \cdot \frac{x-3}{1-3} = \frac{1}{12}(x+2)(x-3)$

   $p(x) = p_0 + p_1 + p_2 = \ldots = -\frac{1}{2}x^2$, so $p(2) = -2$.

2. Using the regula falsi, approximate the root of $f(x) = x^3 - x^2 + 2$ using the initial interval $[-2, 2]$.

   Iteration 1: $c = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(-2)\cdot 6 + 2\cdot 10}{6+10} = \frac{1}{2}$, so $f(c) = 1\frac{7}{8}$.
   Iteration 2: $f(c)f(x_0) = 1\frac{7}{8} \cdot (-10) < 0$, so our new interval of search is $x_0 = -2$ and $x_1 = c = \frac{1}{2}$. The new c now is: $c = \frac{(-2)1\frac{7}{8} - \frac{1}{2}\cdot(-10)}{1\frac{7}{8} - (-10)} \approx 0.105$.
   Succesive values for c upon iteration are approximately:
   $\{0.5, 0.105263, -0.24416, -0.5277, -0.72799, -0.85241, \ldots\}$. The series converges to $c = -1$, we need 20 iterations to obtain 5 decimals of precision.

3. Using Picard iteration approximate the root of $f(x) = x + \cos x$. Start with $x_0 = 0$.

   We need to rewrite $f(x) = x + \cos x = 0$ to a form which says $g(x) = x$, so we need to re-work the equation to something with x on the right side:
   $x + \cos x = 0$
   $\cos x = -x$
   $-\cos x = x$, so $g(x) = -\cos x$.
   The iteration scheme then is:
   $g_0 = -\cos 0 = -1$
   $g_1 = -\cos(-1) \approx -0.5403$
   $g_2 = -\cos(-0.54\ldots) \approx -0.85755$.
   The series converges to ≈ -0.73909. We need 37 iterations to reach 5 decimals of precision.

4. Using the Newton-Raphson method approximate the root of $f(x) = x + \cos x$. Start with $x_0 = 0$.

   The same problem as 3., but now $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x+\cos x}{1-\sin x}$. The iteration scheme then is:
   $g_0 = 0 - \frac{0+1}{1} = -1$
   $g_1 = -1 - \frac{-1+\cos -1}{1-\sin -1} \approx -0.75036$.
   The series converges as in ex.3, but we need only 4 iterations (instead of 37!) to reach 5 decimals of precision.

5. Using the Newton-Raphson method, find the approximation of the root of the function:

   a.) $f(x) = \sin(x) - \cos(x)$
   Similar to ex.4, $g(x) = x - \frac{\sin x - \cos x}{\cos x + \sin x}$, $g_0 = 0 - \frac{0-1}{1+0} = 1$, etc.

b.) $f(x) = x - e^{-x}$

Similar to ex.4, $g(x) = x - \frac{x-e^{-x}}{1+e^{-x}}$, $g_0 = 0 - \frac{0-1}{1+1} = \frac{1}{2}$, etc.

6. Using the first three iterations of the Newton-Raphson method, approximate $\sqrt[3]{2}$. Start with the approximation $\sqrt[3]{2} \approx 2$

   First we need an equation that has $\sqrt[3]{2}$ for a root, *e.g.* $f(x) = x^3 - 2 = 0$. Then, $g(x) = x - \frac{x^3-2}{3x^2}$.
   Starting with $x_0 = 2$, we get
   $g_0 = 2 - \frac{8-2}{12} = \frac{3}{2}$
   $g_1 = \frac{3}{2} - \frac{\frac{27}{8}-2}{3 \cdot \frac{9}{4}} = \frac{35}{27}$
   $g_2 \approx 1.261\ldots$.
   The series converges to $1.259921\ldots$, which is the correct answer.